Proximal Calculus and Universal Feedback Strategies in Two Person Non-Zero Sum Differential Games*

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Abstract

In this paper we introduce the discontinuous universal feedback for the problem of Nash equilibrium in two person non-zero sum differential game. We assume that there exist functions satisfying some conditions analogous to the infinitesimal conditions on value function in zero sum differential games. Under this assumption we prove the existence of universal feedback Nash equilibrium.
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1 Introduction

In this paper we introduce the discontinuous universal feedback for the problem of Nash equilibrium in two person non-zero sum differential game. This approach is close to the extremal shift rule suggested by N.N. Krasovskii and A.I. Subbotin for the zero-sum differential games. The extremal shift rule was suggested in [1] to prove the existence of value function. The value function of zero sum differential game is the viscosity solution of corresponding Hamilton-Jacobi equation. The universal feedback synthesis is based on the properties of value function and the definition of viscosity solutions. N.N. Krasovskii designed the universal feedback ε-strategies [2]. His scheme use the minimization of the value function of the game in ε-neighborhood. A.I. Subbotin introduced the universal feedback using the notion of quasigradient [3], [4]. In the paper of F. H. Clarke, Yu. S. Ledyaev, and A. I. Subbotin [5] the aiming in the direction of proximal subgradients of the value function was studied. Constructed strategy is universal also.

The problem of Nash equilibrium in the differential game is connected with the Cauchy problem for the system of Hamilton-Jacobi equations [6]. In the general case the Hamiltonians are discontinuous. If the smooth solution of Cauchy problem exists and the controls of players are continuous then there exists universal feedback Nash equilibrium. If the solutions doesn't exist or it isn't smooth then the situation becomes more complicated. A particular case when the universal feedback Nash equilibrium exists were considered by P.

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Cardaliaguet and S. Plaskacz [7]. The approach based on the solution of the system of conservation laws is developed by A. Bressan and W. Shen [8], [9]. They study one dimensional games under condition of hyperbolicity and design the universal feedback strategies.

In this paper we consider the general case of finite horizon non-zero sum differential game. We assume that there exist functions satisfying some conditions analogous to the infinitesimal conditions on value function in zero sum differential games [4], [5]. Under this assumption we prove the existence of universal feedback Nash equilibrium.

2 Definitions and Designations

We consider the doubly controlled system

$$\dot{x} = f(t, x, u, v), \quad t \in [t_0, \vartheta_0], \quad x \in \mathbb{R}^n, \quad u \in P, \quad v \in Q. \tag{1}$$

Here u and v are controls of the Player I and the Player II respectively. The purposes of the Players are nonantagonistic. The Player I wants to maximize the functional

$$\Lambda_1(x(\cdot), u(\cdot), v(\cdot)) = \sigma_1(x(\vartheta_0)) + \int_t^{\vartheta_0} g_1(\xi, x(\xi), u(\xi), v(\xi))d\xi,$$

the Player II wants to maximize the functional

$$\Lambda_2(x(\cdot), u(\cdot), v(\cdot)) = \sigma_2(x(\vartheta_0)) + \int_t^{\vartheta_0} g_2(\xi, x(\xi), u(\xi), v(\xi))d\xi.$$

We assume that the sets P and Q are compacts, the function f, σ_1 and σ_2 are continuous, moreover f, g_1 and g_2 are Lipschitz continuous with respect to the phase variable, and satisfy the sublinear growth condition with respect to x.

We use the discontinuous feedback control scheme first suggested by N.N. Krasovskii for zero-sum differential games. We consider the two cases:

- the Players choose feedback strategies and make consistent the corrective moments;
- one of the Players deviates.

Feedback strategy of the Player I is a function $U(t, x)$ with values in P, feedback strategy of the Player II is a function $V(t, x)$ with the values in Q.

Let us consider the first case. We assume that the Player I chooses the strategy U, the Player II chooses the strategy V. Let (t_*, x_*) be an initial position. Suppose that the Players choose the partion of time segment $[t_*, \vartheta_0]$ $\Delta = \{\tau_k\}_{k=1}^m$. Further let $d(\Delta)$ denote the fineness of partition Δ. Let $x^c[\cdot, t_*, x_*, U, V, \Delta]$ be an unique solution of the problem

$$x[t] = x[\tau_i] + \int_{\tau_i}^t f(\xi, x[\xi], u(\tau_i, x[\tau_i]), v(\tau_i, x[\tau_i]))d\xi, \quad t \in [\tau_i, \tau_{i+1}), \quad i = \overline{0, m-1}, \quad x[t_*] = x_*.$$

The value of cost functional of i-th Player in this case is equal to

$$\Lambda_i^c(t_*, x_*, U, V, \Delta) = \sigma_i(x^c[\vartheta_0, t_*, x_*, U, V, \Delta]) +$$
$$\sum_{j=0}^{m-1} \int_{\tau_j}^{\tau_{j+1}} g_i(t, x^c[t, t_*, x_*, U, V, \Delta], u(\tau_i, x^c[\tau_j, t_*, x_*, U, V, \Delta]), v(\tau_i, x^c[\tau_j, t_*, x_*, U, V, \Delta])dt.$$

Now we suppose that the Player II chooses a measurable control $v[\cdot]$. Let $x^1[\cdot, t_*, x_*, U, \Delta, v[\cdot]]$ be an unique solution of the problem

$$x[t] = x[\tau_i] + \int_{\tau_i}^t f(\xi, x[\xi], u(\tau_i, x[\tau_i]), v[\xi])d\xi, \quad t \in [\tau_i, \tau_{i+1}), \quad i = \overline{0, m-1}, \quad x[t_*] = x_*.$$

The value of cost functional of Player II in this case is equal to

$$\Lambda_2^d(t_*, x_*, U, \Delta, v[\cdot]) = \sigma_i(x^1[\vartheta_0, t_*, x_*, U, \Delta, v[\cdot]])+ \\ \sum_{j=0}^{m-1} \int_{\tau_j}^{\tau_{j+1}} g_i(t, x^1[t, t_*, x_*, U, \Delta, v[\cdot]], u(\tau_i, x^c[\tau_j, t_*, x_*, U, \Delta, v[\cdot]]), v(t)dt.$$

In the same way the case when the Player II deviates is considered. Denote the motion generated by the strategy of the Player II and the control of the Player I by $x^2[\cdot, t_*, x_*, V, \Delta, u[\cdot]]$. The value of cost functional of Player I in this case is equal to

$$\Lambda_1^d(t_*, x_*, V, \Delta, u[\cdot]) = \sigma_i(x^2[\vartheta_0, t_*, x_*, V, \Delta, u[\cdot]])+ \\ \sum_{j=0}^{m-1} \int_{\tau_j}^{\tau_{j+1}} g_i(t, x^2[t, t_*, x_*, V, \Delta, u[\cdot]], u[t], x^2[\tau_i, x^2[\tau_j, t_*, x_*, V, \Delta, u[\cdot]])dt.$$

Let us introduce the following values:

$$\Upsilon_i(t_*, x_*, U, V) = \liminf_{d(\Delta)\downarrow 0} \Lambda_i^c(t_*, x_*, U, V, \Delta),$$

$$\Gamma_1(t_*, x_*, V) = \limsup_{d(\Delta)\downarrow 0} \sup_{u[\cdot]} \Lambda_1^d(t_*, x_*, V, \Delta, u[\cdot]),$$

$$\Gamma_2(t_*, x_*, U) = \limsup_{d(\Delta)\downarrow 0} \sup_{v[\cdot]} \Lambda_2^d(t_*, x_*, U, \Delta, v[\cdot]).$$

We say that the family of strategies is universal feedback Nash equilibrium on compact $D_0 \subset [t_0, \vartheta_0] \times \mathbb{R}^n$, if there exists nonnegative functions $\eta_i(\alpha)$, $i = 1, 2$, $\eta_i(\alpha) \to 0 \quad \alpha \to 0$, such that for sufficiently small α the following inequalities are fulfilled for all $(t_*, x_*) \in D_0$

$$\Gamma_1(t_*, x_*, V^\alpha) \le \Upsilon_1(t_*, x_*, U^\alpha, V^\alpha) + \eta_1(\alpha),$$

$$\Gamma_2(t_*, x_*, U^\alpha) \le \Upsilon_2(t_*, x_*, U^\alpha, V^\alpha) + \eta_2(\alpha).$$

In this paper we develop the approach based on system of Hamilton-Jacobi equations. The definition of Hamiltonians involves the Nash equilibrium in a static game. This game is an analog of a small game used in the theory of zero-sum games [1].

Define for $(t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n$, $p, q \in \mathbb{R}^n$, $u \in P$, $v \in Q$ two criterions:

$$\chi_1(t, x, p, q, u, v) \triangleq \langle p, f(t, x, u, v)\rangle + g_1(t, x, u, v),$$

$$\chi_2(t, x, p, q, u, v) \triangleq \langle q, f(t, x, u, v)\rangle + g_2(t, x, u, v).$$

Fix $(t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n$, $p, q \in \mathbb{R}^n$ and consider the static games

$$\begin{cases} \chi_1(t, x, p, q, u, v) \to \max_{u \in P}, \\ \chi_2(t, x, p, q, u, v) \to \max_{v \in Q}. \end{cases} \tag{2}$$

Denote the set of Nash equilibriums of game (2) by $\mathrm{NE}(t, x, p, q)$. Further we assume that for all $(t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n$, $p, q \in \mathbb{R}^n$ the set $\mathrm{NE}(t, x, p, q)$ is nonempty.

Remark 1. If the sets

$$\{\chi_1(t, x, p, q, u, v_*) : u \in P\}, \quad \{\chi_2(t, x, p, q, u_*, v) : v \in Q\}$$

are convex for all $(t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n$, $p, q \in \mathbb{R}^n$, $u_* \in P$, $v_* \in Q$ then $\mathrm{NE}(t, x, p, q)$ is nonempty.

The pair of values $(\chi_1(t, x, p, q, u^*, v^*), \chi_2(t, x, p, q, u^*, v^*))$ for $(u^*, v^*) \in \mathrm{NE}(t, x, p, q)$ is an analog of Hamiltonian.

3 Elements of Proximal Calculus

In this section we follow the definitions from [5]. Let $\phi : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function. The vector $(\zeta_t^-, \zeta_x^-) \in \mathbb{R}^{1+n}$ is said to be a proximal subgradient at the position (t, x), if there exists a constant $\sigma^- > 0$ such that for (t', x') sufficiently close to (t, x) the following inequality holds

$$\phi(t', x') \geq \phi(t, x) + \zeta_t^-(t' - t) + \langle \zeta_x^-, x' - x \rangle - \sigma^- \|(t' - t, x' - x)\|^2.$$

Analogously, the vector $(\zeta_t^+, \zeta_x^+) \in \mathbb{R}^{1+n}$ is said to be a proximal supergradient at the position (t, x), if there exists a constant $\sigma^+ > 0$ such that for (t', x') sufficiently close to (t, x) the following inequality holds

$$\phi(t', x') \leq \phi(t, x) + \zeta_t^+(t' - t) + \langle \zeta_x^+, x' - x \rangle + \sigma^+ \|(t' - t, x' - x)\|^2.$$

Denote the set of all proximal subgradients by $\partial_P^- \phi(t, x)$, the set of all proximal supergradients by $\partial_P^+ \phi(t, x)$.

Let $D_0 \subset [t_0, \vartheta_0] \times \mathbb{R}^n$ be a compact. Denote by D the reachable set from D_0. Let

$$D_1 \triangleq \{(t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n : \|\tau - t, x - y\| \leq 1, \quad (t, x) \in D\}. \tag{3}$$

Let us introduce the following transformation of ϕ:

$$\phi^\alpha(t, x) = \max_{(\tau, y) \in D_1} \Big[\phi(\tau, y) - \frac{1}{2\alpha^2} \|t - \tau, x - y\|^2\Big]. \tag{4}$$

Let (τ^α, y^α) maximize the right hand of (4) for the fixed position $(t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n$. Denote

$$\zeta^\alpha[\phi]_t \triangleq \frac{t - \tau^\alpha}{\alpha^2}, \quad \zeta^\alpha[\phi]_x \triangleq \frac{x - y^\alpha}{\alpha^2}. \tag{5}$$

In the same way we define the following transformation of the function ϕ

$$\phi_\alpha = \min_{(\tau, y) \in D_1} \Big[\phi(\tau, y) + \frac{1}{2\alpha^2} \|t - \tau, x - y\|^2\Big]. \tag{6}$$

Let (τ_α, y_α) minimize the right hand of (6). Denote

$$\zeta_\alpha[\phi]_t \triangleq \frac{t - \tau_\alpha}{\alpha^2}, \quad \zeta_\alpha[\phi]_x \triangleq \frac{x - y_\alpha}{\alpha^2}. \tag{7}$$

Lemmas 1–5 formulated below are analogs of lemmas 3.1–3.5 of [5]. Therefore lemmas 1–5 are not proved here. In lemmas 1–5 we assume that the position $(t, x) \in D$ is fixed, (τ^α, y^α) maximizes the right hand of (4), and (τ_α, y_α) minimizes the right hand of (4).

Lemma 1. *Let (τ_α, y_α) be an inner point of D_1, then*

$$(\zeta_\alpha[\phi]_t, \zeta_\alpha[\phi]_x) \in \partial_P^- \phi(\tau^\alpha, x^\alpha).$$

Analogously, if (τ^α, y^α) is an inner point of D_1, then

$$(\zeta^\alpha[\phi]_t, \zeta^\alpha[\phi]_x) \in \partial_P^+ \phi(\tau^\alpha, x^\alpha).$$

Let

$$k_1 \triangleq \min\{\phi(t, x) : (t, x) \in D_1\}, \quad k_2 \triangleq \max\{\phi(t, x) : (t, x) \in D_1\},$$

$$C_2 \triangleq \sqrt{2(k_2 - k_1)}. \tag{8}$$

Lemma 2. *The following estimates are valid*

$$\|(t-\tau_\alpha, x-y_\alpha)\| \le C_2\alpha, \quad \|(t-\tau^\alpha, x-y^\alpha)\| \le C_2\alpha.$$

Lemma 3. *The following inequalities hold*

$$\frac{\|x-y^\alpha\|^2}{2\alpha^2} \le \omega_\phi(C_2\alpha), \quad \frac{\|x-y^\alpha\|^2}{2\alpha^2} \le \omega_\phi(C_2\alpha).$$

Here ω_ϕ is a modulus of continuity of the function ϕ on D_1.

Following [5] we consider the sets

$$F^\alpha[\phi] = \{(t,x) \in D : \tau^\alpha = \vartheta_0\}, \tag{9}$$

$$F_\alpha[\phi] = \{(t,x) \in D : \tau_\alpha = \vartheta_0\}. \tag{10}$$

Lemma 4. *Let $\alpha \in (0, 1/C_2)$. Then one of the following statements are fulfilled:*

- $(\zeta_\alpha[\phi]_t, \zeta_\alpha[\phi]_x) \in \partial_P^- \phi(\tau_\alpha, y_\alpha)$;
- $(t,x) \in F_\alpha$ *and* $|\vartheta_0 - \tau^\alpha| \le C_2\alpha$.

Analogously, one of the following statements is fulfilled:

- $(\zeta^\alpha[\phi]_t, \zeta^\alpha[\phi]_x) \in \partial_p^+ \phi(\tau^\alpha, y^\alpha)$;
- $(t,x) \in F^\alpha$ *and* $|\vartheta_0 - \tau^\alpha| \le C_2\alpha$.

Lemma 5. *For any $\delta > 0$ and $f \in \mathbb{R}^n$ the following inequalities hold*

$$\phi_\alpha(t+\delta, x+\delta f) \le \phi_\alpha(t,x) + \delta(\zeta_\alpha[\phi]_t + \langle \zeta_\alpha[\phi]_x, f\rangle) + \frac{\delta^2}{2\alpha^2}(1+\|f\|^2),$$

$$\phi^\alpha(t+\delta, x+\delta f) \ge \phi^\alpha(t,x) + \delta(\zeta^\alpha[\phi]_t + \langle \zeta^\alpha[\phi]_x, f\rangle) - \frac{\delta^2}{2\alpha^2}(1+\|f\|^2).$$

4 Main result

Consider the functions $(t,x,p,q) \mapsto u^*(t,x,p,q)$, $(t,x,p,q) \mapsto v^*(t,x,p,q)$ such that $(u^*(t,x,p,q), v^*(t,x,p,q)) \in \mathrm{NE}(t,x,p,q)$. Put

$$\mathcal{H}_i(t,x,p,q) \triangleq \chi_1(t,x,p,q,u^*(t,x,p,q), v^*(t,x,p,q)), \quad i=1,2.$$

Theorem. *Suppose that there exist functions $\varphi(t,x)$, $\psi(t,x)$, and $\omega(t,x,d)$ with properties:*

1. $\omega \ge 0$, $\omega(t,x,d) \to 0$ *as* $d \to 0$ *uniformly on any compact D, $(t,x) \in D$.*
2. $\varphi(\vartheta_0, \cdot) = \sigma_1(\cdot)$;
3. $\psi(\vartheta_0, \cdot) = \sigma_2(\cdot)$;
4. $a^+ + \mathcal{H}_1(t,x,p^+,q^+) \ge -\omega(t,x,d)$ *for all* $(a^+,p^+) \in \partial_P^+\varphi(\theta,z)$, $(b^+,q^+) \in \partial_P^+\psi(\tau,y)$, *such that* $\|(\tau-t, y-x)\|, \|(\theta-t, z-x)\| \le d$;
5. $b^+ + \mathcal{H}_2(t,x,p^+,q^+) \ge -\omega(t,x,d)$ *for all* $(a^+,p^+) \in \partial_P^+\varphi(\theta,z)$, $(b^+,q^+) \in \partial_P^+\psi(\tau,y)$, *such that* $\|(\tau-t, y-x)\|, \|(\theta-t, z-x)\| \le d$;
6. $a^- + \mathcal{H}_1(t,x,p^-,q^+) \le \omega(t,x,d)$ *for all* $(a^+,p^+) \in \partial_P^+\varphi(\theta,z)$, $(b^+,q^+) \in \partial_P^+\psi(\tau,y)$, *such that* $\|(\tau-t, y-x)\|, \|(\theta-t, z-x)\| \le d$;

7. $b^- + \mathcal{H}_2(\tau, y, p^+, q^-) \leq \omega(t, x, \tau, y)$ *for all* $(a^+, p^+) \in \partial_P^+\varphi(\theta, z)$, $(b^+, q^+) \in \partial_P^+\psi(\tau, y)$, *such that* $\|(\tau - t, y - x)\|, \|(\theta - t, z - x)\| \leq d$;

Then the pair of strategies $U^\alpha(t, x) = u^*(t, x, \zeta^\alpha[\varphi]_x, \zeta^\alpha[\psi]_x)$, $V^\alpha(t, x) = v^*(t, x, \zeta^\alpha[\varphi]_x, \zeta^\alpha[\psi]_x)$ *is Nash equilibrium on each compacts* D_0. *Here* $(\zeta[\varphi]_t, \zeta[\varphi]_x)$, $(\zeta[\psi]_t, \zeta[\psi]_x)$ *are defined by formula (5) at the position* (t, x) *for the functions* φ *and* ψ *respectively. Moreover*

$$\liminf_{\alpha\to 0} \Upsilon_1(t, x, U^\alpha, V^\alpha) = \varphi(t, x),$$

$$\liminf_{\alpha\to 0} \Upsilon_2(t, x, U^\alpha, V^\alpha) = \psi(t, x).$$

Corollary. *Suppose that one can choose the functions* $u^*(t, x, p, q)$ *and* $v^*(t, x, p, q)$ *such that its dependence on* (t, x) *is continuous. Moreover we assume that there exists a continuous function* $\varphi(t, x)$, $\psi(t, x)$, *and* $\omega(t, x, d)$ *such that the conditions 1–3 of the Theorem hold and the following conditions are fulfilled:*

4′. $a^+ + \mathcal{H}_1(t, x, p^+, q^+) \geq -\omega(t, x, d)$ *for all* $(a^+, p^+) \in \partial_P^+\varphi(t, x)$, $(b^+, q^+) \in \partial_P^+\psi(\tau, y)$, *such that* $\|(\tau - t, y - x)\| \leq d$;

5′. $b^+ + \mathcal{H}_2(t, x, p^+, q^+) \geq -\omega(t, x, d)$ *for all* $(a^+, p^+) \in \partial_P^+\varphi(t, x)$, $(b^+, q^+) \in \partial_P^+\psi(\tau, y)$, *such that* $\|(\tau - t, y - x)\| \leq d$;

6′. $a^- + \mathcal{H}_1(t, x, p^-, q^+) \leq \omega(t, x, d)$ *for all* $(a^+, p^+) \in \partial_P^+\varphi(t, x)$, $(b^+, q^+) \in \partial_P^+\psi(\tau, y)$, *such that* $\|(\tau - t, y - x)\| \leq d$;

7′. $b^- + \mathcal{H}_2(\tau, y, p^+, q^-) \leq \omega(t, x, \tau, y)$ *for all* $(a^+, p^+) \in \partial_P^+\varphi(t, x)$, $(b^+, q^+) \in \partial_P^+\psi(\tau, y)$, *such that* $\|(\tau - t, y - x)\| \leq d$;

Then the conclusion of the Theorem holds.

Proof of the Theorem. Let us consider the compact D_0. Denote by D the reachable set from D_0. Also we assume that D_1 is defined by (3).

Put

$$F_* = \{(\vartheta_0, x) : x \in \mathbb{R}^n\} \cup F^\alpha[\psi] \cup F_\alpha[\varphi].$$

Let $(t_*, x_*) \in D_0$ be an initial position. Let $\Xi = \{\xi_j\}_{j=0}^r$ be a partition of the interval $[t_*, \vartheta_0]$. Further we will consider a motion of the system $x[\cdot]$, either $x[\cdot] = x[\cdot, t_*, x_*, V, \Xi, u[\cdot]]$, or $x[\cdot] = x[\cdot, t_*, x_*, U, V, \Xi]$. For simplification we put $x_j = x[\xi_j]$.

Let a constant C_3 be define by the rule

$$C_3 \triangleq \max\{\|f(t, x, u, v)\| : (t, x) \in D_1, u \in P, v \in Q\}.$$

Also define

$$C_* \triangleq \sqrt{1 + C_3^2},$$

$$C_4 \triangleq \{|g_1(t, x, u, v)| : (t, x) \in D_1, u \in P, v \in Q\}.$$

Consider the following moduli of continuity on D_1:

$$\omega_f(\delta) \triangleq \{\|f(t', x', u, v) - f(t'', x'', u, v)\| : \\ (t', x'), (t'', x'') \in D_1, \quad u \in P, v \in Q, \quad |t' - t''| \leq \delta, \|x' - x''\| \leq C_3\delta\},$$

$$\omega_{g_1}(\delta) \triangleq \{\|g_1(t', x', u, v) - g_1(t'', x'', u, v)\| : \\ (t', x'), (t'', x'') \in D_1 \quad u \in P, v \in Q, \quad |t' - t''| \leq \delta, \|x' - x''\| \leq C_3\delta\}.$$

First we consider the case then the Player I deviates. We prove that for any control of the Player I $u[\cdot]$, the following inequality is valid

$$\Lambda_1^d(t_*, x_*, V, \Xi, u[\cdot]) \leq \varphi(t_*, x_*) + \eta(\alpha, d(\Xi)), \tag{11}$$

where

$$\lim_{\alpha\to 0}\lim_{\delta\to 0}\eta(\alpha,\delta) = 0.$$

In this case we have $x[\cdot] = x[\cdot, t_*, x_*, V^\alpha, \Xi, u(\cdot)]$. Let l be a minimal number such that $(\xi_l, x[\xi_l]) \in F_*$. Then by lemma 4 we have that $|\xi_l - \theta_0| \leq C_2\alpha$. Hence, $\|x[\vartheta_0] - x[\xi_l]\| \leq C_3\alpha$.

Then

$$\sigma_1(x[\vartheta_0]) = \varphi(\vartheta_0, x[\vartheta_0]) \leq \varphi(\xi_l, x[\xi_l]) - \sum_{k=l}^{r-1}\int_{\xi_k}^{\xi_{k+1}} g_1(\xi, x[\xi], u[\xi], v_k)d\xi + \omega_\varphi(C_*\alpha) + C_4\alpha.$$

We obtain the inequality

$$\sigma_1(x[\vartheta_0]) + \sum_{k=l}^{r-1}\int_{\xi_k}^{\xi_{k+1}} g_1(\xi, x[\xi], u[\xi], v_k)d\xi \leq \varphi(\xi_l, x[\xi_l]) + \omega_\varphi(C_*\alpha) + C_4\alpha. \tag{12}$$

Now let $j < l$. Denote

$$\hat{f} \triangleq \frac{1}{\xi_{j+1} - \xi_j}\int_{\xi_j}^{\xi_{j+1}} f(\xi, x[\xi], u[\xi], v_j).$$

Put $u_j \triangleq U^\alpha(\xi_j, x[\xi_j])$, $v_j \triangleq V^\alpha(\xi_j, x[\xi_j])$. Denote the position (τ_α, y_α) for the function φ and $(\xi_j, x[\xi_j])$ by (τ_1^-, y_1^-). Denote the corresponding pair $(\zeta_\alpha[\varphi]_t, \zeta_\alpha[\varphi]_x)$ by (a^-, p^-). Analogously denote by (τ_1^+, y_1^+) the position (τ^α, y^α) for φ and $(\xi_j, x[\xi_j])$. Also denote the corresponding pair $(\zeta^\alpha[\varphi]_t, \zeta^\alpha[\varphi]_x)$ by (a^+, p^+). Let the position (τ^α, y^α) for ψ and $(\xi_j, x[\xi_j])$ be denoted by (τ_2^+, y_2^+), the corresponding pair $(\zeta^\alpha[\varphi]_t, \zeta^\alpha[\varphi]_x)$ be denoted (b^+, q^+).

By lemma 3 we have inequalities

$$\|p^-\|, \|p^+\| \leq \omega_\varphi(C_2\alpha).$$

By lemma 5 we obtain that

$$\varphi_\alpha(\xi_{j+1}, x[\xi_{j+1}]) \leq \varphi_\alpha(\xi_j, x[\xi_j]) + (\xi_{j+1} - \xi_j)(a^- + \langle p^-, \hat{f}\rangle) + \frac{(\xi_{j+1} - \xi_j)^2}{\alpha^2}(1 + \|\hat{f}\|^2) =$$
$$\varphi_\alpha(\xi_j, x[\xi_j]) + (\xi_{j+1} - \xi_j)a^- + \int_{\xi_j}^{\xi_{j+1}} \langle p^-, f(\xi, x[\xi], u[\xi], v_j)\rangle d\xi + \frac{(\xi_{j+1} - \xi_j)^2}{\alpha^2}(1 + \|\hat{f}\|^2) =$$
$$\varphi_\alpha(\xi_j, x[\xi_j]) + (\xi_{j+1} - \xi_j)a^- + \int_{\xi_j}^{\xi_{j+1}} \langle p^-, f(\xi_j, x[\xi_j], u[\xi], v_j)\rangle d\xi +$$
$$(\xi_{j+1} - \xi_j)\omega_\varphi(C_2\alpha)\omega_f(\xi_{j+1} - \xi_j) + \frac{(\xi_{j+1} - \xi_j)^2}{\alpha^2}(1 + \|\hat{f}\|^2) \leq$$
$$\varphi_\alpha(\xi_j, x[\xi_j]) + (\xi_{j+1} - \xi_j)a^- + (\xi_{j+1} - \xi_j)\max_{u\in P}\Big[\langle p^-, f(\xi_j, x[\xi_j], u, v_j)\rangle + g_1(\xi_j, x[\xi_j], u, v_j)\Big] -$$
$$\int_{\xi_j}^{\xi_j} g_1(\xi, x[\xi], u[\xi], v_j)d\xi + (\xi_{j+1} - \xi_j)\omega_{g_1}(\xi_{j+1} - \xi_j) +$$
$$(\xi_{j+1} - \xi_j)\omega_\varphi(C_2\alpha)\omega_f(\xi_{j+1} - \xi_j) + \frac{(\xi_{j+1} - \xi_j)^2}{\alpha^2}(1 + \|\hat{f}\|^2). \tag{13}$$

By the choice of v_j we have that

$$a^- + \max_{u\in P}\Big[\langle p^-, f(\xi_j, x[\xi_j], u, v_j)\rangle + g_1(\xi_j, x[\xi_j], u, v_j)\Big] = a^- + \mathcal{H}_1(\xi_j, x[\xi_j], p^-, q^+). \quad (14)$$

It follows from the properties $(a^-, p^-) \in \partial_P^- \varphi(\tau_1^-, y_1^-)$, $(a^-, p^-) \in \partial_P^+ \varphi(\tau_2^+, y_2^+)$,

$$\|(\tau_1^- - \xi_j, y_1^- - x[\xi_j])\|, \|(\tau_2^+ - \xi_j, y_2^+ - x[\xi_j])\| \leq C_2\alpha$$

and condition 6 of the Theorem that

$$a^- + \mathcal{H}_1(t, x, p^-, q^+) \leq \omega(\tau_1^-, y_1^-, C_2\alpha). \quad (15)$$

Put

$$\gamma(\alpha) \triangleq \sup\{\omega(t', x', C_2\alpha) : (t', x') \in D\}.$$

By the condition 1 of the Theorem we have that $\gamma(\alpha) \to 0$, as $\alpha \to 0$. From the inequalities (13), (14) and (15) it follows that

$$\varphi_\alpha(\xi_{j+1}, x[\xi_{j+1}]) + \int_{\xi_j}^{\xi_{j+1}} g_1(\xi, x[\xi], u[\xi], v_j)d\xi \leq$$

$$\varphi_\alpha(\xi_j, x[\xi_j]) + (\xi_{j+1} - \xi_j)\Big[\gamma(\alpha) + \omega_\varphi(C_2\alpha)\omega_f(\xi_{j+1} - \xi_j) + \omega_{g_1}(\xi_{j+1} - \xi_j) + \frac{(\xi_{j+1} - \xi_j)}{\alpha^2}C_*^2\Big]. \quad (16)$$

Using the estimate (16) for $j = \overline{0, l-1}$ we conclude that

$$\varphi_\alpha(\xi_l, x[\xi_l]) + \sum_{k=0}^{l-1}\int_{\xi_k}^{\xi_{k+1}} g_1(\xi, x[\xi], u[\xi], v_k)d\xi \leq$$

$$\varphi_\alpha(t_*, x_*) + (\xi_l - t_*)\Big[\gamma(\alpha) + \omega_\varphi(C_2\alpha)\omega_f(\delta) + \omega_{g_1}(\delta) + \frac{\delta}{\alpha^2}C_*^2\Big].$$

Here $\delta = d(\Xi)$.

Since

$$\varphi_\alpha(\xi_l, x[\xi_l]) \geq \varphi(\xi_l, x[\xi_l]) - \omega_\varphi(\alpha),$$

using the inequality (12) we have that

$$\sigma(x[\vartheta_0]) + \sum_{k=0}^{r-1}\int_{\xi_k}^{\xi_{k+1}} g_1(\xi, x[\xi], u[\xi], v_k)d\xi \leq$$

$$\varphi_\alpha(t_*, x_*) + (\vartheta_0 - t_*)\Big[\gamma(\alpha) + \omega_\varphi(C_2\alpha)\omega_f(\delta) + \omega_{g_1}(\delta) + \frac{\delta}{\alpha^2}C_*^2\Big] + \omega_\varphi(\alpha) + C_4\alpha + \omega_\varphi(\alpha).$$

Denoting

$$\eta(\alpha, \delta) \triangleq (\vartheta_0 - t_0)\Big[\gamma(\alpha) + \omega_\varphi(C_2\alpha)\omega_f(\delta) + \omega_{g_1}(\delta) + \frac{\delta}{\alpha^2}C_*^2\Big] + \omega_\varphi(C_*\alpha) + C_4\alpha + \omega_\varphi(\alpha),$$

we obtain the estimate (11).

Now let us consider the case when the Player I and Player II use the strategies U^α and V^α respectively. Let $x[\cdot]$ denote the motion $x^c[\cdot, t_*, x_*, U, V, \Xi]$.

Put

$$F^* \triangleq \{(\vartheta_0, x) : x \in \mathbb{R}^n\} \cup F^\alpha[\varphi] \cup F^\alpha[\psi].$$

Let l be a minimal number such that $(\xi_l, x[\xi_l]) \in F^*$. By lemma 4, $|\xi_l - \theta_0| \le C_2\alpha$. Therefore, $\|x[\vartheta_0] - x[\xi_l]\| \le C_3\alpha$. Estimating the variation of the function φ we have that

$$\sigma_1(x[\vartheta_0]) = \varphi(\vartheta_0, x[\vartheta_0]) \ge \varphi(\xi_l, x[\xi_l]) - \int_{\xi_l}^{\vartheta_0} g_1(\xi, x[\xi], u^c(\xi,\Xi), v^c[\xi,\Xi])d\xi - \omega_\varphi(C_*\alpha) - C_4\alpha.$$

Hence we conclude

$$\sigma_1(x[\vartheta_0]) + \sum_{k=l}^{r-1} \int_{\xi_k}^{\xi_{k+1}} g_1(\xi, x[\xi], u_k, v_k)d\xi \ge \varphi(\xi_l, x[\xi_l]) - \omega_\varphi(C_*\alpha) - C_4\alpha. \tag{17}$$

Now let $j < l$. Denote

$$\tilde{f} = \frac{1}{\xi_{j+1} - \xi_j} \int_{\xi_j}^{\xi_{j+1}} f(\xi, x[\xi], u_j, v_j)d\xi.$$

As to the rest we use the notations introduced above. By lemma 5 we obtain that

$$\begin{aligned}
\varphi(\xi_{j+1}, x[\xi_{j+1}]) &\ge \varphi(\xi_j, x[\xi_j]) + (\xi_{j+1} - \xi_j)(a^+ + \langle p^+, \tilde{f}\rangle) - \frac{(\xi_{j+1} - \xi_j)^2}{2\alpha^2}(1 + \|\tilde{f}\|^2) = \\
&\varphi(\xi_j, x[\xi_j]) + (\xi_{j+1} - \xi_j)a^+ + \int_{\xi_j}^{\xi_{j+1}} \langle p^+, f(\xi, x[\xi], u_j, v_j)d\xi\rangle)d\xi - \frac{(\xi_{j+1} - \xi_j)^2}{2\alpha^2}(1 + \|\tilde{f}\|^2) \ge \\
&\varphi(\xi_j, x[\xi_j]) + (\xi_{j+1} - \xi_j)[a^+ + \langle p^+, f(\xi_j, x[\xi_j], u_j, v_j)\rangle + g_1(\xi_j, x[\xi_j], u_j, v_j)] - \\
&- \int_{\xi_j}^{\xi_{j+1}} g_1(\xi, x[\xi], u_j, v_j)d\xi - (\xi_{j+1} - \xi_j)(\omega_\varphi(C_2\alpha)\omega_f(\xi_{j+1} - \xi_j) + \omega_{g_1}(\xi_{j+1} - \xi_j)) - \\
&\frac{(\xi_{j+1} - \xi_j)^2}{2\alpha^2}C_*^2.
\end{aligned} \tag{18}$$

It follows from the definition of the strategies U^α, V^α and the elements u_j, v_j that

$$a^+ + \langle p^+, f(\xi_j, x[\xi_j], u_j, v_j)\rangle + g_1(\xi_j, x[\xi_j], u_j, v_j) = a^+ + \mathcal{H}_1(\xi_j, x[\xi_j], p^+, q^+).$$

By the lemma 2 we have that

$$\|\tau_1^+ - \xi_j, y_1^+ - x[\xi_j]\|, \|\tau_2^+ - \xi_j, y_2^+ - x[\xi_j]\| \le C_2\alpha$$

The condition 4 of the Theorem yields that

$$a^+ + \mathcal{H}(t, x, p^+, q^+) \ge -\omega(t, x, C_2\alpha).$$

Using the function $\gamma(\cdot)$ we have

$$a^+ + \mathcal{H}_1(\xi_j, x[\xi_j], p^+, q^+) \ge -\gamma(\alpha).$$

From this and the inequality (18) it follows that

$$\begin{aligned}
&\varphi(\xi_{j+1}, x[\xi_{j+1}]) + \int_{\xi_j}^{\xi_{j+1}} g_1(\xi, x[\xi], u_j, v_j)d\xi \ge \\
&\varphi(\xi_j, x[\xi_j]) - (\xi_{j+1} - \xi_j)\Big[\gamma(\alpha) + \omega_\varphi(C_2\alpha)\omega_f(\xi_{j+1} - \xi_j) + \omega_{g_1}(\xi_{j+1} - \xi_j) + \frac{(\xi_{j+1} - \xi_j)}{2\alpha^2}C_*^2\Big].
\end{aligned} \tag{19}$$

Using the inequalities (19) for $j=\overline{0,l-1}$ we obtain that

$$\varphi(\xi_l, x[\xi_l]) + \sum_{k=0}^{l-1} \int_{\xi_k}^{\xi_{k+1}} g_1(\xi, x[\xi], u_k, v_k) d\xi \geq$$
$$\varphi(t_*, x_*) - (\xi_l - t_*)\Big[\gamma(\alpha) + \omega_\varphi(C_2\alpha)\omega_f(\delta) + \omega_{g_1}(\delta) + \frac{\delta}{2\alpha^2}C_*^2\Big]. \quad (20)$$

Here $\delta = d(\Xi)$. Using the estimate (17) and the property

$$\varphi^\alpha(\xi_l, x[\xi_l]) \leq \varphi(\xi_l, x[\xi_l]) + \omega_\varphi(\alpha),$$

we conclude from (20) that

$$\Lambda_i^c(t_*, x_*, U^\alpha, V^\alpha, \Delta) = \sigma(x[\vartheta_0]) + \sum_{k=0}^{r-1} \int_{\xi_k}^{\xi_{k+1}} g_1(\xi, x[\xi], u_k, v_k) d\xi \geq$$
$$\varphi^\alpha(t_*, x_*) - (\vartheta_0 - t_*)\Big[\gamma(\alpha) + \omega_\varphi(C_2\alpha)\omega_f(\delta) + \omega_{g_1}(\delta) + \frac{\delta}{\alpha^2}C_*^2\Big] - \omega_\varphi(\alpha) - C_4\alpha - \omega_\varphi(\alpha). \quad (21)$$

Letting δ to 0 in (11) and using the definition of η we obtain that

$$\Gamma_1(t_*, x_*, V^\alpha) \leq \varphi(t_*, x_*) + (\vartheta_0 - t_0)\gamma(\alpha) + \omega_\varphi(C_*\alpha) + C_4\alpha + \omega_\varphi(\alpha). \quad (22)$$

In the same manner we obtain from (21) that

$$\Upsilon[t_*, x_*, U^\alpha, V^\alpha] \geq \varphi(t_*, x_*) - (\vartheta_0 - t_*)\gamma(\alpha) - \omega_\varphi(\alpha) - C_4\alpha - \omega_\varphi(\alpha). \quad (23)$$

Therefore

$$\Gamma[t_*, x_*, V^\alpha] - \varkappa(\alpha) \leq \varphi(t_*, x_*) \leq \Upsilon[t_*, x_*, U^\alpha, V^\alpha] + \varkappa(\alpha). \quad (24)$$

Here

$$\varkappa(\alpha) \triangleq (\vartheta_0 - t_*)\gamma(\alpha) + \omega_\varphi(\alpha) + C_4\alpha + \omega_\varphi(\alpha).$$

The analog of the inequality (24) is fulfilled for the function of the Player II. Therefore the pair of strategies U^α, V^α is the universal Nash feedback. Moreover the payoffs of the Players I and the Player II at the position (t_*, x_*) are equal to $\varphi(t_*, x_*)$ and $\psi(t_*, x_*)$ respectively. □

Proof of the Corollary. In this case the inequalities (22) and (23) are still valid if we replace the function $\gamma(\cdot)$ with the function $\tilde\gamma(\cdot)$

$$\tilde\gamma(\alpha) \triangleq \gamma(\alpha) + \omega_\varphi(C_2\alpha)\omega_f^*(C_2\alpha) + \omega_{g_1}^*(C_2\alpha),$$

where ω_f^* is the following modulus of continuity

$$\omega_f^*(\delta) \triangleq \sup\{\|f(t', x', u^*(t', x', p, q), v^*(t', x', p, q)) - f(t'', x'', u^*(t'', x'', p, q), v^*(t'', x'', p, q)) :$$
$$(t', x'), (t'', x'') \in D_1, \|(t' - t'', x' - x'')\| \leq \delta, \|p\| \leq \omega_\varphi(\delta), \|q\| \leq \omega_\psi(\delta)\}.$$

The function $\omega_{g_1}^*(\cdot)$ is defined in the same way. Now let us consider the analog of the inequality (14). We use the designations introduced in the proof of the Theorem. We have that

$$a^- + \max_{u\in P}\Big[\langle p^-, f(\xi_j, x[\xi_j], u, v_j)\rangle + g_1(\xi_j, x[\xi_j], u, v_j)\Big] =$$
$$a^- + \langle p^-, f(\xi_j, x_j, u^*(\xi_j, x_j, p^-, q^+), v^*(\xi_j, x_j, p^-, q^+))\rangle +$$
$$g_1(\xi_j, x_j, u^*(\xi_j, x_j, p^-, q^+), v^*(\xi_j, x_j, p^-, q^+)$$

From the lemma 3 and the definition of the function ω^*_f, we conclude that

$$\begin{aligned}a^- + \langle p^-, f(\xi_j, x_j, u^*(\xi_j, x_j, p^-, q^+), v^*(\xi_j, x_j, p^-, q^+))\rangle +\\ g_1(\xi_j, x_j, u^*(\xi_j, x_j, p^-, q^+), v^*(\xi_j, x_j, p^-, q^+) \leq\\ a^- + \langle p^-, f(\tau_1^-, y_1^-, u^*(\tau_1^-, y_1^-, p^-, q^+), v^*(\tau_1^-, y_1^-, p^-, q^+))\rangle +\\ g_1(\tau_1^-, y_1^-, u^*(\tau_1^-, y_1^-, p^-, q^+), v^*(\tau_1^-, y_1^-, p^-, q^+)) + \omega_\varphi(C_2\alpha)\omega^*_f(C_2\alpha) + \omega^*_{g^1}(C_2\alpha) =\\ a^- + \mathcal{H}_1(\tau_1^-, y_1^-, p^-, q^+) + \omega_\varphi(C_2\alpha)\omega^*_f(C_2\alpha) + \omega^*_{g^1}(C_2\alpha).\end{aligned}$$

Using the condition $6'$ we obtain the analog of the of the inequality (22). The analog of the inequality (23) is obtained in the same way.

□

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